

# On the Dimension of the Stability Group for a Levi Non-Degenerate Hypersurface \* †

V. V. Ezhov and A. V. Isaev

*We classify locally defined Levi non-degenerate non-spherical real-analytic hypersurfaces in complex space for which the dimension of the group of local CR-automorphisms has the second largest positive value.*

## 1 Introduction

Let  $M$  be a real-analytic hypersurface in  $\mathbb{C}^{n+1}$  passing through the origin. Assume that the Levi form of  $M$  at 0 is non-degenerate and has signature  $(n-m, m)$  with  $n \geq 2m$ . Then in some local holomorphic coordinates  $z = (z_1, \dots, z_n)$ ,  $w = u + iv$  in a neighborhood of the origin,  $M$  can be written in the Chern-Moser normal form (see [CM]), that is, given by an equation

$$v = \langle z, z \rangle + \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \bar{z}, u),$$

where  $\langle z, z \rangle = \sum_{\alpha, \beta=1}^n h_{\alpha\beta} z_\alpha \bar{z}_\beta$  is a non-degenerate Hermitian form with signature  $(n-m, m)$ , and  $F_{k\bar{l}}(z, \bar{z}, u)$  are polynomials of degree  $k$  in  $z$  and  $\bar{l}$  in  $\bar{z}$  whose coefficients are analytic functions of  $u$  such that the following conditions hold

$$\begin{aligned} \text{tr } F_{2\bar{2}} &\equiv 0, \\ \text{tr}^2 F_{2\bar{3}} &\equiv 0, \\ \text{tr}^3 F_{3\bar{3}} &\equiv 0. \end{aligned} \tag{1.1}$$

Here the operator  $\text{tr}$  is defined as

$$\text{tr} := \sum_{\alpha, \beta=1}^n \hat{h}_{\alpha\beta} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta},$$

---

\*Mathematics Subject Classification: 32V40

†Keywords and Phrases: Levi non-degenerate hypersurfaces, Chern-Moser normal forms, linearization of local CR-automorphisms.

where  $(\hat{h}_{\alpha\beta})$  is the matrix inverse to  $H := (h_{\alpha\beta})$ . Everywhere below we assume that  $M$  is given in the normal form.

Let  $\text{Aut}_0(M)$  denote the group of all local CR-automorphisms of  $M$  defined near 0 and preserving 0. To avoid confusion with the term “isotropy group of  $M$  at 0” usually reserved for global CR-automorphisms of  $M$  preserving the origin, this group is often called the *stability group* of  $M$  at 0. Every element  $\varphi$  of  $\text{Aut}_0(M)$  extends to a biholomorphic mapping defined in a neighborhood of the origin in  $\mathbb{C}^{n+1}$  and therefore can be written as

$$\begin{aligned} z &\mapsto f_\varphi(z, w), \\ w &\mapsto g_\varphi(z, w), \end{aligned}$$

where  $f_\varphi$  and  $g_\varphi$  are holomorphic. We equip  $\text{Aut}_0(M)$  with the topology of uniform convergence of the partial derivatives of all orders of the component functions on a neighborhood of 0. The group  $\text{Aut}_0(M)$  with this topology is a topological group.

It follows from [CM] that every element  $\varphi = (f_\varphi, g_\varphi)$  of  $\text{Aut}_0(M)$  is uniquely determined by a set of parameters  $(U_\varphi, a_\varphi, \lambda_\varphi, \sigma_\varphi, r_\varphi)$ , where  $\sigma_\varphi = \pm 1$ ,  $U_\varphi$  is an  $n \times n$ -matrix such that  $\langle U_\varphi z, U_\varphi z \rangle = \sigma_\varphi \langle z, z \rangle$  for all  $z \in \mathbb{C}^n$ ,  $a_\varphi \in \mathbb{C}^n$ ,  $\lambda_\varphi > 0$ ,  $r_\varphi \in \mathbb{R}$  (note that  $\sigma_\varphi$  can be equal to  $-1$  only for  $n = 2m$ ). These parameters are determined by the following relations

$$\frac{\partial f_\varphi}{\partial z}(0) = \lambda_\varphi U_\varphi, \quad \frac{\partial f_\varphi}{\partial w}(0) = \lambda_\varphi U_\varphi a_\varphi,$$

$$\frac{\partial g_\varphi}{\partial w}(0) = \sigma_\varphi \lambda_\varphi^2, \quad \text{Re } \frac{\partial^2 g_\varphi}{\partial^2 w}(0) = 2\sigma_\varphi \lambda_\varphi^2 r_\varphi.$$

For results on the dependence of local CR-mappings on their jets in more general settings see [BER1], [BER2], [Eb], [Z].

We assume that  $M$  is *non-spherical at the origin*, i.e., that  $M$  in a neighborhood of the origin is not CR-equivalent to an open subset of the hyperquadric given by the equation  $v = \langle z, z \rangle$ . In this case for every element  $\varphi = (f_\varphi, g_\varphi)$  of  $\text{Aut}_0(M)$  the parameters  $a_\varphi, \lambda_\varphi, \sigma_\varphi, r_\varphi$  are uniquely determined by the matrix  $U_\varphi$ , and the mapping

$$\Phi : \text{Aut}_0(M) \rightarrow GL_n(\mathbb{C}), \quad \Phi : \varphi \mapsto U_\varphi$$

is a topological group isomorphism between  $\text{Aut}_0(M)$  and  $G_0 := \Phi(\text{Aut}_0(M))$  with  $G_0$  being a real algebraic subgroup of  $GL_n(\mathbb{C})$ ; in addition the mapping

$$\Lambda : G_0(M) \rightarrow \mathbb{R}_+, \quad \Lambda : U_\varphi \mapsto \lambda_\varphi \quad (1.2)$$

is a Lie group homomorphism with the property  $\Lambda(U_\varphi) = 1$  if all eigenvalues of  $U_\varphi$  are unimodular, where  $\mathbb{R}_+$  is the group of positive real numbers with respect to multiplication (see [CM], [B], [L1], [BV], [VK]). Since  $G_0(M)$  is a closed subgroup of  $GL_n(\mathbb{C})$ , we can pull back its Lie group structure to  $\text{Aut}_0(M)$  by means of  $\Phi$  (note that the pulled back topology is identical to that of  $\text{Aut}_0(M)$ ). Let  $d_0(M)$  denote the dimension of  $\text{Aut}_0(M)$ . We are interested in characterizing hypersurfaces for which  $d_0(M)$  is large.

If  $n > 2m$ ,  $G_0(M)$  is a closed subgroup of the pseudounitary group  $U(n-m, m)$  of all matrices  $U$  such that

$$U^t H \bar{U} = H,$$

(recall that  $H$  is the matrix of the Hermitian form  $\langle z, z \rangle$ ). The group  $U(n, 0)$  is the unitary group  $U(n)$ . If  $n = 2m$ ,  $G_0$  is a closed subgroup of the group  $U'(m, m)$  of all matrices  $U$  such that

$$U^t H \bar{U} = \pm H,$$

that has two connected components. In particular, we always have  $d_0(M) \leq n^2$ . If  $d_0(M) = n^2$  and  $n > 2m$ , then  $G_0(M) = U(n-m, m)$ . If  $d_0(M) = n^2$  and  $n = 2m$ , then we have either  $G_0(M) = U(m, m)$ , or  $G_0(M) = U'(m, m)$ .

We will say that the group  $\text{Aut}_0(M)$  is *linearizable*, if in some coordinates every  $\varphi \in \text{Aut}_0(M)$  can be written in the form

$$\begin{aligned} z &\mapsto \lambda U z, \\ w &\mapsto \sigma \lambda^2 w. \end{aligned} \tag{1.3}$$

Clearly, in the above formula  $U = U_\varphi$ ,  $\lambda = \lambda_\varphi$ ,  $\sigma = \sigma_\varphi$ . The group  $\text{Aut}_0(M)$  is known to be linearizable, for example, for  $m = 0$  (see [KL]) and for  $m = 1$  (see [Ezh1], [Ezh2]). If all elements of  $\text{Aut}_0(M)$  in some coordinates have the form (1.3), we say that  $\text{Aut}_0(M)$  is *linear* in these coordinates. It is shown in Lemma 3 of [Ezh3] that if  $\text{Aut}_0(M)$  is linear in some coordinates, it is linear in some normal coordinates as well.

We will first discuss the case when  $d_0(M)$  takes the largest possible value, that is, when  $d_0(M) = n^2$ . Observe that in this case  $\text{Aut}_0(M)$  is linearizable for any  $m$ . Indeed, if  $d_0(M) = n^2$ , the group  $G_0(M)$  contains  $U(n-m, m)$ . Hence  $G_0(M)$  contains the subgroup  $Q := \{e^{it} \cdot E_n, t \in \mathbb{R}\}$ , where  $E_n$  is the  $n \times n$  identity matrix. Let  $\hat{Q} = \Phi^{-1}(Q) \subset \text{Aut}_0(M)$ . The subgroup  $\hat{Q}$  is compact, and the argument in [KL] (see also [VK]) yields that in some normal coordinates every element of  $\hat{Q}$  can be written in the form (1.3). For every  $\varphi \in \hat{Q}$  we clearly have  $\sigma_\varphi = 1$ . Further, since  $Q$  is

compact, there are no non-trivial homomorphisms from  $Q$  into  $\mathbb{R}_+$ , and therefore  $\lambda_\varphi = 1$  for every  $\varphi \in \hat{Q}$ . Hence, in these coordinates the function

$$F(z, \bar{z}, u) := \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \bar{z}, u)$$

is invariant under all linear transformations from  $Q$  and thus  $F_{k\bar{l}} \equiv 0$ , if  $k \neq \bar{l}$ . We will now show that  $\text{Aut}_0(M)$  is linearizable. Since linearizability arguments of this kind will occur several times throughout the paper, we give some details on the linearizability of  $\text{Aut}_0(M)$  for general hypersurfaces.

Suppose that  $M$  is given in the Chern-Moser normal form near the origin. The main step in showing that  $\text{Aut}_0(M)$  is linearizable is to prove that in some normal coordinates for every  $\varphi \in \text{Aut}_0(M)$ , we have  $a_\varphi = 0$ . Indeed, if  $a_\varphi = 0$ , it follows from [CM] that  $\varphi$  in the given coordinates is a fractional linear transformation that becomes linear if  $r_\varphi = 0$ . It is shown in the proof of Proposition 3 of [L2] that  $a_\varphi = 0$  implies that  $r_\varphi = 0$ , provided  $\lambda_\varphi = 1$ . Furthermore, if for every  $\varphi \in \text{Aut}_0(M)$  we have  $a_\varphi = 0$  and there exists  $\varphi_0 \in \text{Aut}_0(M)$  with  $\lambda_{\varphi_0} \neq 1$ , the group  $\text{Aut}_0(M)$  becomes linear after applying a transformation of the form

$$\begin{aligned} z &\mapsto \frac{z}{1 + qw}, \\ w &\mapsto \frac{w}{1 + qw}, \end{aligned} \tag{1.4}$$

for some  $q \in \mathbb{R}$ .

To prove that  $a_\varphi = 0$  for a fixed  $\varphi = (f_\varphi, g_\varphi)$  in the given coordinates, we introduce weights as follows. Let each of  $z_1, \dots, z_n$ ,  $\bar{z}_1, \dots, \bar{z}_n$  be of weight 1 and  $u$  be of weight 2. Then we can write a weight decomposition for the function  $F$  as follows

$$F(z, \bar{z}, u) = \sum_{j=\gamma}^{\infty} F_j,$$

where  $F_j$  is the component of  $F$  of weight  $j$ , and  $F_\gamma \not\equiv 0$ . Next, since  $\varphi$  is a local automorphism of  $M$ , we have

$$\text{Im } g_\varphi = \langle f_\varphi, f_\varphi \rangle + F(f_\varphi, \bar{f}_\varphi, \text{Re } g_\varphi), \tag{1.5}$$

where we set  $v = \langle z, z \rangle + F(z, \bar{z}, u)$ . Extracting all terms of weight  $\gamma + 1$  from identity (1.5), we obtain the following identity (see [B], [L1], [L2])

$$\begin{aligned} &\text{Re} \left( i\tilde{g}_{\gamma+1} + 2\langle \lambda_\varphi^{-1} U_\varphi^{-1} \tilde{f}_\gamma, z \rangle \right) |_{v=\langle z, z \rangle} + T(F_\gamma, a_\varphi) = \\ &F_{\gamma+1}(z, \bar{z}, u) - \frac{1}{\lambda_\varphi^2} F_{\gamma+1}(\lambda_\varphi U_\varphi z, \overline{\lambda_\varphi U_\varphi z}, \lambda_\varphi^2 u). \end{aligned} \tag{1.6}$$

Here  $(\sum_{j=1}^{\infty} \tilde{f}_j, \sum_{j=1}^{\infty} \tilde{g}_j)$  is the weight decomposition for the map  $(\tilde{f}, \tilde{g}) := (f_{\varphi} - f_{\varphi}^Q, g_{\varphi} - g_{\varphi}^Q)$ , where  $\varphi^Q = (f_{\varphi}^Q, g_{\varphi}^Q)$  is the following local automorphism of the hyperquadric given by the equation  $v = \langle z, z \rangle$

$$\begin{aligned} z &\mapsto \frac{\lambda_{\varphi} U_{\varphi}(z + a_{\varphi} w)}{1 - 2i\langle z, a \rangle - (r_{\varphi} + i\langle a, a \rangle)w}, \\ w &\mapsto \frac{\sigma_{\varphi} \lambda_{\varphi}^2 w}{1 - 2i\langle z, a \rangle - (r_{\varphi} + i\langle a, a \rangle)w}, \end{aligned}$$

and

$$\begin{aligned} T(F_{\gamma}, a_{\varphi}) := 2\operatorname{Re} \Big( &-2i\langle z, a_{\varphi} \rangle F_{\gamma} + (u + i\langle z, z \rangle) \sum_{j=1}^n a_j \frac{\partial F_{\gamma}}{\partial z_j} + \\ &2i\langle z, a \rangle \sum_{j=1}^n z_j \frac{\partial F_{\gamma}}{\partial z_j} + i\langle z, a_{\varphi} \rangle (u + i\langle z, z \rangle) \frac{\partial F_{\gamma}}{\partial u} \Big), \end{aligned}$$

where  $a_1, \dots, a_n$  denote the components of the vector  $a_{\varphi}$ .

If  $F_{\gamma+1} = 0$ , the right-hand side of (1.6) vanishes, and the proof of Proposition 1 in [L1] shows that the resulting homogeneous identity can only hold if  $a_{\varphi} = 0$ . Clearly, if  $F_{k\bar{l}} \equiv 0$  for  $k \neq \bar{l}$ , then the weight decomposition for  $F$  contains only terms of even weights, and, in particular, we have  $F_{\gamma+1} = 0$ . Thus, we have shown that  $\operatorname{Aut}_0(M)$  is linearizable if  $d_0(M) = n^2$ .

Observe further that if  $d_0(M) = n^2$ , the mapping  $\Lambda$  defined in (1.2) is constant, that is,  $\lambda_{\varphi} = 1$  for all  $\varphi \in \operatorname{Aut}_0(M)$ . Indeed, consider the restriction of  $\Lambda$  to  $U(n-m, m)$ . Every element  $U \in U(n-m, m)$  can be represented as  $U = e^{i\psi} \cdot V$  with  $\psi \in \mathbb{R}$  and  $V \in SU(n-m, m)$ . Note that there are no non-trivial homomorphisms from the unit circle into  $\mathbb{R}_+$  since  $\mathbb{R}_+$  has no non-trivial compact subgroups. Also, there are no non-trivial homomorphisms from  $SU(n-m, m)$  into  $\mathbb{R}_+$  since the kernel of any such homomorphism is a proper normal subgroup of  $SU(n-m, m)$  of positive dimension, and  $SU(n-m, m)$  is a simple group. Thus,  $\Lambda$  is constant on  $U(n-m, m)$  and hence on all of  $G_0(M)$ . It then follows that, in coordinates in which  $\operatorname{Aut}_0(M)$  is linear, the function  $F$  is invariant under all linear transformations of the  $z$ -variables from  $U(n-m, m)$  and therefore depends only on  $\langle z, z \rangle$  and  $u$ . Conditions (1.1) imply that  $F_{2\bar{2}} \equiv 0$ ,  $F_{3\bar{3}} \equiv 0$ . Thus,  $F$  has the form

$$F(z, \bar{z}, u) = \sum_{k=4}^{\infty} C_k(u) \langle z, z \rangle^k, \quad (1.7)$$

where  $C_k(u)$  are real-valued analytic functions of  $u$ , and for some  $k$  we have  $C_k(u) \not\equiv 0$ . Note, in particular, that if  $d_0(M) = n^2$ , then 0 is an umbilic point of  $M$ .

Conversely, if  $M$  is given in the normal form by an equation

$$v = \langle z, z \rangle + F(z, \bar{z}, u),$$

with  $F \not\equiv 0$  of the form (1.7), then  $\text{Aut}_0(M)$  contains all linear transformations (1.3) with  $U \in U(n-m, m)$ ,  $\lambda = 1$  and  $\sigma = 1$ , and therefore  $d_0(M) = n^2$ . For  $n > 2m$  and for  $n = 2m$  with  $G_0(M) = U(m, m)$ ,  $\text{Aut}_0(M)$  clearly coincides with the group of all transformations of the form

$$\begin{aligned} z &\mapsto Uz, \\ w &\mapsto w. \end{aligned} \tag{1.8}$$

where  $U \in U(n-m, m)$ . If  $n = 2m$  and  $G_0(M) = U'(m, m)$ , then  $\text{Aut}_0(M)$  consists of all mappings

$$\begin{aligned} z &\mapsto Uz, \\ w &\mapsto \sigma w, \end{aligned}$$

where  $U \in U'(m, m)$ ,  $\langle Uz, Uz \rangle = \sigma \langle z, z \rangle$ ,  $\sigma = \pm 1$ .

We will now concentrate on the case  $0 < d_0(M) < n^2$  (hence assuming that  $n \geq 2$ ). For the strongly pseudoconvex case we obtain the following

**THEOREM 1.1** *Let  $M$  be a strongly pseudoconvex real-analytic non-spherical hypersurface in  $\mathbb{C}^{n+1}$  with  $n \geq 2$  (here  $m = 0$ ) given in normal coordinates in which  $\text{Aut}_0(M)$  is linear. Then the following holds*

(i)  $d_0(M) \geq n^2 - 2n + 3$  implies  $d_0(M) = n^2$ ;

(ii) if  $d_0(M) = n^2 - 2n + 2$ , after a linear change of the  $z$ -coordinates the equation of  $M$  takes the form

$$v = \sum_{\alpha=1}^n |z_\alpha|^2 + F(z, \bar{z}, u), \tag{1.9}$$

where  $F$  is a function of  $|z_1|^2$ ,  $\langle z, z \rangle := \sum_{\alpha=1}^n |z_\alpha|^2$  and  $u$ :

$$F(z, \bar{z}, u) = \sum_{p+q \geq 4} C_{pq}(u) |z_1|^{2p} \langle z, z \rangle^q, \tag{1.10}$$

where  $C_{pq}(u)$  are real-valued analytic functions of  $u$ , and  $C_{pq}(u) \not\equiv 0$  for some  $p, q$  with  $p > 0$ ;

(iii) if a hypersurface  $M$  is given in the form described in (ii) (without assuming the linearity of  $\text{Aut}_0(M)$  a priori), the group  $\text{Aut}_0(M)$  coincides with the group of all mappings of the form (1.8), where  $U \in U(1) \times U(n-1)$  (with  $U(1) \times U(n-1)$  realized as a group of block-diagonal matrices in the standard way).

**Corollary 1.2** If  $M$  is a strongly pseudoconvex real-analytic hypersurface in  $\mathbb{C}^{n+1}$ , and the dimension of  $\text{Aut}_0(M)$  is greater than or equal to  $n^2 - 2n + 2$ , then the origin is an umbilic point of  $M$ .

For the case  $m \geq 1$  we prove the following

**THEOREM 1.3** Let  $M$  be a Levi non-degenerate real-analytic non-spherical hypersurface in  $\mathbb{C}^{n+1}$  with  $m \geq 1$ . Then the following holds

(i)  $d_0(M) \geq n^2 - 2n + 4$  implies  $d_0(M) = n^2$ ;

(ii) if  $d_0(M) = n^2 - 2n + 3$ , the group  $\text{Aut}_0(M)$  is linearizable and in some normal coordinates in which  $\text{Aut}_0(M)$  is linear, the equation of  $M$  takes the form

$$\begin{aligned} v &= 2\text{Re } z_1\bar{z}_n + 2\text{Re } z_2\bar{z}_{n-1} + \dots + 2\text{Re } z_m\bar{z}_{n-m+1} + \\ &\quad \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2 + F(z, \bar{z}, u), \end{aligned} \tag{1.11}$$

where  $F$  is a function of  $|z_n|^2$ ,  $\langle z, z \rangle := 2\text{Re } z_1\bar{z}_n + 2\text{Re } z_2\bar{z}_{n-1} + \dots + 2\text{Re } z_m\bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2$  and  $u$ :

$$F(z, \bar{z}, u) = \sum C_{rpq} u^r |z_n|^{2p} \langle z, z \rangle^q, \tag{1.12}$$

where at least one of  $C_{rpq} \in \mathbb{R}$  is non-zero, the summation is taken over  $p \geq 1$ ,  $q \geq 0$ ,  $r \geq 0$  such that  $(r+q-1)/p = s$  with  $s \geq -1/2$  being a fixed rational number, and

$$F(z, \bar{z}, u) = \sum_{k, l \geq 2} F_{kl}(z, \bar{z}, u),$$

where  $F_{2\bar{3}} = 0$  and identities (1.1) hold for  $F_{2\bar{2}}$  and  $F_{3\bar{3}}$ ;

(iii) if a hypersurface  $M$  is given in the form described in (ii) (without assuming the linearity of  $\text{Aut}_0(M)$  a priori), the group  $\text{Aut}_0(M)$  coincides with the group of all mappings of the form

$$\begin{aligned} z &\mapsto |\mu|^{1/(s+1)}Uz, \\ w &\mapsto |\mu|^{2/(s+1)}w, \end{aligned} \quad (1.13)$$

with  $U \in S$ , where  $S$  is the group introduced in Lemma 3.1 below, and  $\mu$  is a parameter in this group (see formula (3.2)).

**Corollary 1.4** *Let  $M$  be a Levi non-degenerate real-analytic hypersurface in  $\mathbb{C}^{n+1}$ , with  $n \geq 2$  and  $m \geq 1$ , and assume that the dimension of  $\text{Aut}_0(M)$  is greater than or equal to  $n^2 - 2n + 3$ . If the origin is a non-umbilic point of  $M$ , then in some normal coordinates the equation of  $M$  takes the form*

$$\begin{aligned} v = & 2\operatorname{Re} z_1\bar{z}_n + 2\operatorname{Re} z_2\bar{z}_{n-1} + \dots + \\ & 2\operatorname{Re} z_m\bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2 \pm |z_n|^4. \end{aligned} \quad (1.14)$$

We remark that hypersurfaces (1.14) occur in [P] in connection with studying unbounded homogeneous domains in complex space.

The proofs of Theorems 1.1 and 1.3 are given in Sections 2 and 3 respectively. Before proceeding we wish to acknowledge that this work was initiated while the first author was visiting the Mathematical Sciences Institute of the Australian National University.

## 2 The Strongly Pseudoconvex Case

First of all, we note that in this case the mapping  $\Lambda$  defined in (1.2) is constant, that is,  $\lambda_\varphi = 1$  for all  $\varphi \in \text{Aut}_0(M)$ . This follows from the fact that all eigenvalues of  $U_\varphi$  are unimodular, or, alternatively, from the compactness of  $G_0(M)$  and the observation that  $\mathbb{R}_+$  does not have non-trivial compact subgroups. Next, by a linear change of the  $z$ -coordinates the matrix  $H$  can be transformed into the identity matrix  $E_n$ , and for the remainder of this section we assume that  $H = E_n$ . Hence we assume that the equation of  $M$  is written in the form (1.9), where the function  $F$  satisfies the normal form conditions.

It is shown in Lemma 2.1 of [IK] that any closed connected subgroup of the unitary group  $U(n)$  of dimension  $n^2 - 2n + 3$  or larger is either  $SU(n)$  or  $U(n)$  itself. Hence, if  $d_0(M) \geq n^2 - 2n + 3$ , we have  $G_0(M) \supset SU(n)$ , and therefore  $F(z, \bar{z}, u)$  is invariant under all linear transformations of the  $z$ -variables from  $SU(n)$ . This

implies that  $F(z, \bar{z}, u)$  is a function of  $\langle z, z \rangle$  and  $u$ , which gives that  $F(z, \bar{z}, u)$  is invariant under the action of the full unitary group  $U(n)$  and thus  $d_0(M) = n^2$ , as stated in (i).

The proof of part (ii) of the theorem is also based on Lemma 2.1 of [IK]. For the case  $d_0(M) = n^2 - 2n + 2$  the lemma gives that the connected identity component  $G_0^c$  of  $G_0$  is either conjugate in  $U(n)$  to the subgroup  $U(1) \times U(n-1)$  realized as block-diagonal matrices, or, for  $n = 4$ , contains a subgroup conjugate in  $U(n)$  to  $Sp_{2,0}$ . If the latter is the case, then, since  $Sp_{2,0}$  acts transitively on the sphere of dimension 7 in  $\mathbb{C}^4$ ,  $F(z, \bar{z}, u)$  is a function of  $\langle z, z \rangle$  and  $u$ , which implies that  $F(z, \bar{z}, u)$  is invariant under the action of the full unitary group  $U(4)$  and thus  $d_0(M) = 16$ , which is impossible. Hence  $G_0^c$  is conjugate to  $U(1) \times U(n-1)$ , and therefore, after a unitary change of the  $z$ -coordinates, the equation of  $M$  can be written in the form (1.9) where the function  $F$  depends on  $|z_1|^2$ ,

$$\langle z, z \rangle' := \sum_{\alpha=2}^n |z_\alpha|^2 \text{ and } u. \quad \text{Clearly, } \langle z, z \rangle' = \langle z, z \rangle - |z_1|^2, \text{ and } F$$

can be written as a function of  $|z_1|^2$ ,  $\langle z, z \rangle$  and  $u$  as in (1.10). Next, conditions (1.1) imply that  $F_{2\bar{2}} \equiv 0$ ,  $F_{3\bar{3}} \equiv 0$ , and thus the summation in (1.10) is taken over  $p, q$  such that  $p+q \geq 4$ . Further, if  $C_{pq} \equiv 0$  for all  $p > 0$ ,  $F$  has the form (1.7) and therefore  $G_0 = U(n)$  which is impossible. Thus for some  $p, q$  with  $p > 0$  we have  $C_{pq} \not\equiv 0$ , and (ii) is established.

If  $M$  is given in the normal form and is written as in (1.9), (1.10),  $\text{Aut}_0(M)$  clearly contains all maps of the form (1.8) with  $U \in U(1) \times U(n-1)$ . Hence  $d_0(M) \geq n^2 - 2n + 2$ . If  $d_0(M) > n^2 - 2n + 2$ , then by part (i) of the theorem,  $d_0(M) = n^2$  and hence  $G_0(M) = U(n)$ . Then  $F$  has the form (1.7) which is impossible because for some  $p, q$  with  $p > 0$  the function  $C_{pq}$  does not vanish identically. Thus  $d_0(M) = n^2 - 2n + 2$ , and hence  $G_0^c(M) = U(1) \times U(n-1)$ . It is not hard to show that  $G_0(M)$  is connected (note, for example, that by an argument given in the introduction,  $\text{Aut}_0(M)$  is linear in these coordinates), and therefore  $\text{Aut}_0(M)$  coincides with the group of all mappings of the form (1.8), where  $U \in U(1) \times U(n-1)$ .

Thus, (iii) is established, and the theorem is proved.  $\square$

### 3 The Case of $m \geq 1$

We start with the following algebraic lemma.

**Lemma 3.1** *Let  $G \subset U(n-m, m)$  be a connected real algebraic subgroup of  $GL_n(\mathbb{C})$ ,  $n \geq 2m$ ,  $m \geq 1$ , with Hermitian form pre-*

served by  $U(n-m, m)$  written as

$$\begin{pmatrix} & & & 1 \\ 0 & & & \ddots \\ & & 1 & \\ & & E_{n-2m} & \\ & 1 & & \\ & \ddots & & 0 \\ 1 & & & \end{pmatrix}, \quad (3.1)$$

where  $E_{n-2m}$  is the  $(n-2m) \times (n-2m)$  identity matrix, and the number of 1's on each side of  $E_{n-2m}$  is  $m$ . Then the following holds

(a) if  $\dim G \geq n^2 - 2n + 4$ , we have either  $G = SU(n-m, m)$ , or  $G = U(n-m, m)$ ;

(b) if  $\dim G = n^2 - 2n + 3$ , the group  $G$  either is conjugate in  $U(n-m, m)$  to the group  $S$  that consists of all matrices of the form

$$\begin{pmatrix} \mu & -\mu \bar{x}^T H' A & c \\ 0 & A & x \\ 0 & 0 & 1/\bar{\mu} \end{pmatrix}, \quad (3.2)$$

where  $\mu, c \in \mathbb{C}$ ,  $\mu \neq 0$ ,  $x \in \mathbb{C}^{n-2}$ ,  $A \in U(n-m-1, m-1)$  (i.e.,  $A$  is an  $(n-2) \times (n-2)$ -matrix with complex elements such that  $A^T H' \bar{A} = H'$  with  $H'$  obtained from matrix (3.1) by removing the first and the last columns and rows), and the following holds

$$2\operatorname{Re} \frac{c}{\mu} + x^T H' \bar{x} = 0,$$

or, if  $n = 4$  and  $m = 2$ , coincides with  $e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$ , or, if  $n = 2$  and  $m = 1$ , coincides with  $SU(1, 1)$ . Here the subgroup  $Sp_4(B, \mathbb{C}) \subset GL_4(\mathbb{C})$  consists of matrices preserving a non-degenerate skew-symmetric bilinear form  $B$  equivalent to the form given by the matrix

$$B_0 := \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}, \quad (3.3)$$

where  $E_2$  is the  $2 \times 2$  identity matrix,

**Proof:** Let  $V \subset U(n-m, m)$  be a real algebraic subgroup of  $GL_n(\mathbb{C})$  such that  $\dim V \geq n^2 - 2n + 3$ . Consider  $V_1 := V \cap SU(n-m, m)$ . Clearly,  $\dim V_1 \geq n^2 - 2n + 2$ . Let  $V_1^\mathbb{C} \subset SL_n(\mathbb{C})$  be the complexification of  $V_1$ . We have  $\dim_{\mathbb{C}} V_1^\mathbb{C} \geq n^2 - 2n + 2$ . Consider the maximal complex closed subgroup  $W(V) \subset SL_n(\mathbb{C})$  that contains  $V_1^\mathbb{C}$ . Clearly,  $\dim_{\mathbb{C}} W(V) \geq n^2 - 2n + 2$ . All closed maximal subgroups of  $SL_n(\mathbb{C})$  had been classified (see [D]), and the lower bound on the dimension of  $W(V)$  gives that either  $W(V) = SL_n(\mathbb{C})$ , or  $W(V)$  is conjugate to one of the parabolic subgroups

$$P^1 := \left\{ \begin{pmatrix} 1/\det C & b \\ 0 & C \end{pmatrix}, b \in \mathbb{C}^{n-1}, C \in GL_{n-1}(\mathbb{C}) \right\},$$

$$P^2 := \left\{ \begin{pmatrix} C & b \\ 0 & 1/\det C \end{pmatrix}, b \in \mathbb{C}^{n-1}, C \in GL_{n-1}(\mathbb{C}) \right\}$$

(note that  $P^1 = P^2$  for  $n = 2$ ), or, for  $n = 4$ ,  $W(V)$  is conjugate to  $Sp_4(\mathbb{C})$ .

Suppose that for some  $g \in SL_n(\mathbb{C})$  and  $j \in \{1, 2\}$  we have  $g^{-1}W(V)g = P^j$ . It is not hard to show that, due to the lower bound on the dimension of  $W(V)$ ,  $g$  can be chosen to belong to  $SU(n-m, m)$ . Then  $g^{-1}V_1g \subset P^j \cap SU(n-m, m)$ . It is easy to compute the intersections  $P^j \cap SU(n-m, m)$  for  $j = 1, 2$  and see that they are equal and coincide with the group  $S_1$  of matrices of the form (3.2) with determinant 1. Clearly,  $\dim S_1 = n^2 - 2n + 2 \leq \dim V_1$  and therefore  $V_1$  is conjugate to  $S_1$  in  $SU(n-m, m)$ .

Suppose now that  $n = 4$  and for some  $g \in SL_4(\mathbb{C})$  we have  $g^{-1}W(V)g = Sp_4(\mathbb{C})$ . In particular,  $g^{-1}V_1g \subset Sp_4(\mathbb{C}) \cap g^{-1}SU(4-m, m)g$  (here we have either  $m = 1$ , or  $m = 2$ ). It can be shown that  $\dim Sp_4(\mathbb{C}) \cap g^{-1}SU(3, 1)g \leq 6$  for all  $g \in SL_4(\mathbb{C})$ . At the same time we have  $\dim V_1 \geq 10$ . Hence  $W(V)$  in fact cannot be conjugate to  $Sp_4(\mathbb{C})$ , if  $m = 1$ . Therefore,  $m = 2$ , and  $V_1 \subset gSp_4(\mathbb{C})g^{-1} \cap SU(2, 2) = Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ , where  $B$  is some non-degenerate skew-symmetric bilinear form. It is straightforward to show that  $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$  is connected and  $\dim Sp_4(B, \mathbb{C}) \cap SU(2, 2) \leq 10$ . Therefore  $V_1 = Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ .

Suppose now that  $\dim G \geq n^2 - 2n + 4$ . Then  $\dim G_1 \geq n^2 - 2n + 3$ , and the above considerations give that  $W(G) = SL_n(\mathbb{C})$ . Hence  $G_1 = SU(n-m, m)$  which implies that either  $G = SU(n-m, m)$ , or  $G = U(n-m, m)$ , thus proving (a).

Let  $\dim G = n^2 - 2n + 3$ . In this case we have either  $\dim G_1 = n^2 - 2n + 2$ , or  $G = SU(1, 1)$ , if  $n = 2, m = 1$ . In the first case we obtain that  $G_1$  either is conjugate to  $S_1$  in  $SU(n-m, m)$ , or,

for  $n = 4$  and  $m = 2$  coincides with  $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$  for some non-degenerate skew-symmetric bilinear form  $B$  equivalent to the form  $B_0$  defined in (3.3). This gives that  $G$  in the first case either is conjugate to  $S$  in  $U(n-m, m)$ , or for  $n = 4$  and  $m = 2$  coincides with  $e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$ , and (b) is established.

The lemma is proved.  $\square$

We will now prove Theorem 1.3. Suppose first that  $d_0(M) \geq n^2 - 2n + 4$  and assume that  $H$  is written in the diagonal form with 1's in the first  $n-m$  positions and  $-1$ 's in the last  $m$  positions on the diagonal. Lemma 3.1 gives that either  $G_0^c(M) = SU(n-m, m)$  (in which case  $n \geq 3$ ), or  $G_0(M) = U(n-m, m)$ , or, for  $n = 2m$ ,  $G_0(M) = U'(m, m)$ . If  $G_0(M) \supset U(n-m, m)$ , then  $d_0(M) = n^2$ , and (i) is established. Assume that  $G_0(M) \supset SU(n-m, m)$ . Suppose that  $m \geq 2$ . Then  $G_0$  contains the product  $R := SU(n-m) \times SU(m)$  realized as block-diagonal matrices. Arguing as in the introduction, we obtain that in some normal coordinates all elements of the compact group  $\hat{R} := \Phi^{-1}(R)$  can be written in the form (1.8) and thus  $F$  is a function of  $\langle z, z \rangle_+ := \sum_{j=1}^{n-m} |z_j|^2$ ,  $\langle z, z \rangle_- := \sum_{j=n-m+1}^n |z_j|^2$ , and  $u$ . Hence all elements of odd weight in the weight decomposition for  $F$  are zero. This shows that  $F_{\gamma+1} \equiv 0$ , and identity (1.6) again implies that  $\text{Aut}_0(M)$  becomes linear after a change of coordinates of the form (1.4). If  $m = 1$ ,  $\text{Aut}_0(M)$  is linearizable by [Ezh1], [Ezh2].

Therefore, there exist normal coordinates where the corresponding function  $F$  is invariant under all linear transformations of the  $z$ -variables from  $SU(n-m, m)$ . This implies that  $F$  is in fact invariant under all linear transformations of the  $z$ -variables from  $U(n-m, m)$ . Hence  $d_0(M) = n^2$ , and (i) is established.

Suppose now that  $d_0(M) = n^2 - 2n + 3$ . By a linear change of the  $z$ -coordinates the matrix  $H$  can be transformed into matrix (3.1), and from now on we assume that  $H$  is given in this form. Hence the equation of  $M$  is written as in (1.11), where the function  $F$  satisfies the normal form conditions. Arguing as in the preceding paragraph, we see that for  $n = 2$ ,  $m = 1$ , the group  $G_0^c$  cannot coincide with  $SU(1, 1)$ . Assume first that after a linear change of the  $z$ -coordinates preserving the form  $H$  the group  $G_0^c(M)$  coincides with  $S$ . Then  $G_0(M)$  contains the compact subgroup  $Q = \{e^{it} \cdot E_n, t \in \mathbb{R}\}$ , where  $E_n$  is the  $n \times n$  identity matrix. The argument based on identity (1.6) that we gave in the introduction, again yields that  $\text{Aut}_0(M)$  is linearizable. Passing to coordinates in which  $\text{Aut}_0(M)$  is linear, we obtain that for every  $U \in S$  the equation of

$M$  is invariant under the linear transformation

$$\begin{aligned} z &\mapsto \lambda_U U z, \\ w &\mapsto \lambda_U^2 w, \end{aligned} \tag{3.4}$$

where  $\lambda_U = \Lambda(U)$ . The group  $S$  contains  $U(n-m-1, m-1)$  realized as the subgroup of all matrices of the form (3.2) with  $\mu = 1$ ,  $c = 0$ ,  $x = 0$ . Since  $\Lambda$  is constant on  $U(n-m-1, m-1)$ , we have  $\lambda_U = 1$  for all  $U \in U(n-m-1, m-1)$ . Therefore, the function  $F(z, \bar{z}, u)$  depends on  $z_1, z_n, \bar{z}_1, \bar{z}_n, \langle z, z \rangle' := 2\operatorname{Re} z_2 \bar{z}_{n-1} + \dots + 2\operatorname{Re} z_m \bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2$  and  $u$ . Clearly,  $\langle z, z \rangle' = \langle z, z \rangle - 2\operatorname{Re} z_1 \bar{z}_n$ , and  $F$  can be written as follows

$$F(z, \bar{z}, u) = \sum_{r,q \geq 0} D_{rq}(z_1, z_n, \bar{z}_1, \bar{z}_n) u^r \langle z, z \rangle^q,$$

where  $D_{rq}$  are real-analytic.

We will now determine the form of the functions  $D_{rq}$ . The group  $S$  contains the subgroup  $I$  of all matrices as in (3.2) with  $|\mu| = 1$ ,  $x = 0$  and  $A = E_{n-2}$ , where  $E_{n-2}$  is the  $(n-2) \times (n-2)$  identity matrix. Since every eigenvalue of any  $U \in I$  is unimodular, we have  $\lambda_U = 1$  for all  $U \in I$ , and therefore  $D_{rq}$  is invariant under all linear transformations from  $I$ . It is straightforward to show (see also [Ezh2]) that any polynomial of  $z_1, z_n, \bar{z}_1, \bar{z}_n$  invariant under all linear transformations from  $I$  is a function of  $\operatorname{Re} z_1 \bar{z}_n$  and  $|z_n|^2$ , and hence every  $D_{rq}$  has this property. Let further  $J$  be the subgroup of  $S$  given by the conditions  $\mu = 1$ ,  $A = E_{n-2}$ . For every  $U \in J$  we also have  $\lambda_U = 1$ , and hence  $D_{rq}$  is invariant under all linear transformations from  $J$ . It is then easy to see that  $D_{rq}$  has to be a function of  $|z_n|^2$  alone. Thus, the function  $F$  has the form (1.12), and it remains to show that the summation in (1.12) is taken over  $p \geq 1, q \geq 0, r \geq 0$  such that  $(r+q-1)/p = s$ , where  $s \geq -1/2$  is a fixed rational number.

Let  $K$  be the 1-dimensional subgroup of  $S$  given by the conditions  $\mu > 0, c = 0, x = 0, A = E_{n-2}$ . It is straightforward to show that every homomorphism  $\Psi : K \rightarrow \mathbb{R}_+$  has the form  $U \mapsto \mu^\alpha$ , where  $\alpha \in \mathbb{R}$ . Considering  $\Psi = \Lambda|_K$  we obtain that there exists  $\alpha \in \mathbb{R}$  such that for every  $U \in K$  we have  $\lambda_U = \mu^\alpha$ . We will now prove that  $\alpha \neq 0$ . Indeed, otherwise  $F$  would be invariant under all linear transformations from  $K$  and therefore would be a function of  $\langle z, z \rangle$  and  $u$ , which implies that  $G_0(M) \supset U(n-m, m)$ . This contradiction shows that  $\alpha \neq 0$  and hence  $\lambda_U \neq 1$  for every  $U \in K$  with  $\mu \neq 1$ .

Plugging a mapping of the form (3.4) with  $U \in K$ ,  $\mu \neq 1$ , into equation (1.11), where  $F \not\equiv 0$  has the form (1.12) we obtain that, if  $C_{rpq} \neq 0$ , then

$$\lambda_U^{r+p+q-1} = \mu^p. \quad (3.5)$$

The equation of  $M$  is written in the normal form, hence  $p + q \geq 2$  and  $r + p + q - 1 \geq 1$ . Since  $\lambda_U \neq 1$ , we obtain that  $p \geq 1$ . Further, (3.5) implies

$$\lambda_U^{(r+p+q-1)/p} = \mu,$$

and, since the right-hand side in the above identity does not depend on  $r, p, q$ , for all non-zero coefficients  $C_{rpq}$  the ratio  $(r + q - 1)/p$  must have the same value; we denote it by  $s$ . Clearly,  $s$  is a rational number and  $s \geq -1/2$ . We also remark that  $\alpha = p/(r + p + q - 1) = 1/(s + 1)$ .

Assume now that  $n = 4$ ,  $m = 2$  and  $G_0^c(M)$  coincides with  $e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$  for some non-degenerate skew-symmetric non-degenerate bilinear form  $B$  equivalent to the form  $B_0$  defined in (3.3). Then  $G_0(M)$  contains the compact subgroup  $Q = \{e^{it} \cdot E_4, t \in \mathbb{R}\}$ , where  $E_4$  is the  $4 \times 4$  identity matrix. Arguing as above, we obtain that  $\text{Aut}_0(M)$  is linearizable. Further, it is straightforward to prove that  $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$  is a real form of  $Sp_4(B, \mathbb{C})$  and therefore is simple. Hence there does not exist a non-trivial homomorphism from  $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$  into  $\mathbb{R}_+$ . Further, since  $\mathbb{R}_+$  does not have non-trivial compact subgroups, any homomorphism from the unit circle into  $\mathbb{R}_+$  is constant. Hence  $\Lambda$  is constant on  $G_0(M)$ . This implies that  $F$  is invariant under all linear transformations from  $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ . It can be shown that this group acts transitively on any pseudosphere in  $\mathbb{C}^4$  given by the equation  $\langle z, z \rangle = r$ , which yields that  $F$  is a function of  $\langle z, z \rangle$  and  $u$  and hence  $d_0(M) = n^2$ . This contradiction proves that in fact  $G_0^c(M) \neq e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$  for  $n = 4$ ,  $m = 2$ . Thus, (ii) is established.

Suppose that  $M$  is given in the normal form, written as in (1.11), (1.12), and the summation in (1.12) is taken over  $p \geq 1$ ,  $q \geq 0$ ,  $r \geq 0$  such that  $(r + q - 1)/p = s$ , where  $s \geq -1/2$  is a fixed rational number. Set  $\alpha = 1/(s + 1)$  and for every  $U \in S$  define  $\lambda_U = |\mu|^\alpha$ . It is then straightforward to verify that every mapping of the form (3.4) with  $U \in S$  is an automorphism of  $M$ . Therefore,  $G_0(M)$  contains  $S$  and hence  $d_0(M) \geq n^2 - 2n + 3$ . If  $d_0(M) > n^2 - 2n + 3$ , then by part (i) of the theorem,  $d_0(M) = n^2$  and hence  $G_0(M) \supset U(n-m, m)$ . Then  $F$  is a function of  $\langle z, z \rangle$  and  $u$ , which is impossible since for every non-zero  $C_{rpq}$  we have  $p \geq 1$ . Hence  $d_0(M) = n^2 - 2n + 3$  and hence  $G_0^c(M) = S$ . Finally, observe that

by an argument given in the introduction,  $\text{Aut}_0(M)$  is linear in these coordinates. It is now straightforward to show that  $\text{Aut}_0(M)$  coincides with the group of all mappings of the form (1.13).

Thus, (iii) is established, and the theorem is proved.  $\square$

## References

- [BER1] Baouendi, M. S., Ebenfelt, P., Rothschild, L. P., Rational dependence of smooth and analytic CR mappings on their jets, *Math. Ann.* 315(1999), 205–249.
- [BER2] Baouendi, M. S., Ebenfelt, P., Rothschild, L. P., CR automorphisms of real analytic manifolds in complex space, *Comm. Anal. Geom.* 6(1998), 291–315.
- [B] Beloshapka, V. K., On the dimension of the group of automorphisms of an analytic hypersurface (translated from Russian), *Math. USSR-Izv.* 14(1980), 223–245.
- [BV] Beloshapka, V.K. and Vitushkin, A.G., Estimates for the radius of convergence of power series defining mappings of analytic hypersurfaces (translated from Russian), *Math. USSR-Izv.* 19(1982), 241–259.
- [CM] Chern, S. S. and Moser, J. K., Real hypersurfaces in complex manifolds, *Acta Math.* 133(1974), 219–271.
- [D] Dynkin, E. B., Maximal subgroups of the classical groups (translated from Russian), *Amer. Math. Soc. Translations II, Ser. 6* (1957), 245–378.
- [Eb] Ebenfelt, P., Finite jet determination of holomorphic mappings at the boundary, *Asian J. Math.* 5(2001), 637–662.
- [Ezh1] Ezhov, V. V., Linearization of stability group for a class of hypersurfaces (translated from Russian), *Russian Math. Surveys* 41(1986), 203–204.
- [Ezh2] Ezhov, V. V., On the linearization of automorphisms of a real analytic hypersurface (translated from Russian), *Math. USSR-Izv.* 27(1986), 53–84.
- [Ezh3] Ezhov, V. V., Example of a real-analytic hypersurface with nonlinearizable stability group (translated from Russian), *Math. Notes* 44(1988), 824–828.

- [IK] Isaev, A. V. and Krantz, S. G., On the automorphism groups of hyperbolic manifolds, *J. Reine Angew. Math.* 534(2001), 187–194.
- [KL] Kruzhilin, N. G. and Loboda, A.B., Linearization of local automorphisms of pseudoconvex surfaces (translated from Russian), *Sov. Math., Dokl.* 28(1983), 70–72.
- [L1] Loboda, A. V., On local automorphisms of real-analytic hypersurfaces (translated from Russian), *Math. USSR-Izv.* 18(1982), 537–559.
- [L2] Loboda, A.V., Linearizability of automorphisms of non-spherical surfaces (translated from Russian), *Math. USSR-Izv.* 21(1983), 171–186.
- [P] Penney, R., Homogeneous Koszul manifolds in  $\mathbb{C}^n$ , *J. Diff. Geom.* 36(1992), 591–631.
- [VK] Vitushkin, A. G. and Kruzhilin, N. G., Description of the automorphism groups of real hypersurfaces of complex space (Russian), Investigations in the theory of the approximation of functions, Akad. Nauk SSSR Bashkir. Filial, Otdel Fiz. Mat., Ufa, 1987, 26–69.
- [Z] Zaitsev, D., Unique determination of local CR-maps by their jets: a survey, in Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001), *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 13(2002), 295–305.

School of Mathematics and Statistics  
 University of South Australia  
 Mawson Lakes Blvd  
 Mawson Lakes  
 South Australia 5091  
 AUSTRALIA  
 E-mail: vladimir.ejov@unisa.edu.au

Department of Mathematics  
 The Australian National University  
 Canberra, ACT 0200  
 AUSTRALIA  
 E-mail: alexander.isaev@maths.anu.edu.au